The Bernoulli numbers

Xavier Gourdon and Pascal Sebah
April 24, 2000[1]

1 Introduction

The *Bernoulli numbers* plays an important and quite mysterious role in mathematics, in various places like analysis, number theory and differential topology. They first appeared in *Ars conjectandi*, a (posthumous) published treatise in 1713, by Jakob Bernoulli (1654-1705) when he studied the sums of powers of consecutive integers

$$s_p(n) = \sum_{k=1}^{n-1} k^p, \tag{1}$$

where p and n two given positive integers. Bernoulli's numbers also appear in the computation of

$$\zeta(2p) = \sum_{k=1}^{\infty} \frac{1}{k^{2p}}$$

and in the expansion of many usual functions as $\tan(x)$, $\tanh(x)$, $1/\sin(x)$, $\cdots$ Perhaps one of the most important result is *Euler-Maclaurin* summation formula, where Bernoulli's numbers are contained and which allows to accelerate the computation of slow converging series (see the essay on Euler's constant). They also appear in numbers theory (Fermat's theorem) and in many other domains.

According to Louis Saalschültz [12], the term *Bernoulli's numbers* was used for the first time by Abraham De Moivre (1667-1754) and Leonhard Euler (1707-1783).

1.1 Bernoulli's approach

In its first integral computations of the function $x \mapsto x^p$, Pierre de Fermat (1601-1665) in 1636 had to evaluate the sums $s_p(n)$ defined by (1) and in 1631, Johann Faulhaber (1580-1635) computed explicit formulas for these sums up to $p = 17$ (read the excellent [9] for the beginnings of integration). Thus, it was already known to Jakob Bernoulli that

$$\begin{aligned} s_0(n) &= n \\ s_1(n) &= \frac{1}{2}n^2 - \frac{1}{2}n = \frac{n(n-1)}{2} \\ s_2(n) &= \frac{1}{3}n^3 - \frac{1}{2}n^2 + \frac{1}{6}n = \frac{n(n-1)(2n-1)}{6} \end{aligned}$$

[1] This pages are from *//xavier.gourdon.free.fr/Constants/constants.html*

$$\begin{aligned} s_3(n) &= \frac{1}{4}n^4 - \frac{1}{2}n^3 + \frac{1}{4}n^2 = \frac{n^2(n-1)^2}{4} \\ s_4(n) &= \frac{1}{5}n^5 - \frac{1}{2}n^4 + \frac{1}{3}n^3 - \frac{1}{30}n = \frac{n(n-1)(2n-1)(3n^2-3n-1)}{30} \\ s_5(n) &= \frac{1}{6}n^6 - \frac{1}{2}n^5 + \frac{5}{12}n^4 - \frac{1}{12}n^2 = \frac{n^2(2n^2-2n-1)(n-1)^2}{12} \\ &\dots \end{aligned}$$

Jakob Bernoulli then noticed empirically that the polynomials $s_p(n)$ have the form

$$s_p(n) = \frac{1}{p+1}n^{p+1} - \frac{1}{2}n^p + \frac{p}{12}n^{p-1} + 0 \cdot n^{p-2} + \dots$$

In this expression, the numbers $(1, -1/2, 1/12, 0, ...)$ are appearing and do not depend on p. More generally, the sums $s_p(n)$ can be written in the form

$$\begin{aligned} s_p(n) &= \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k} \\ &= \frac{B_0}{0!}\frac{n^{p+1}}{p+1} + \frac{B_1}{1!}n^p + \frac{B_2}{2!}pn^{p-1} + \frac{B_3}{3!}p(p-1)n^{p-2} + \dots + \frac{B_p}{1!}n \end{aligned}$$

where the B_k are numbers which are independent of p and called *Bernoulli numbers.*

We have

$$B_0 = 1, B_1 = -\frac{1}{2}, B_2 = \frac{1}{6}, B_3 = 0, ...$$

To illustrate the usefulness of his formula, Bernoulli computed the astonishing value of $s_{10}(1000)$ with little effort (in less than "half a quarter of an hour" he says ...)

$$91409924241424243424241924242500$$

(you can check it !). To achieve this he needed to find B_0 up to B_{10}.

There is good evidence that the famous Japanese mathematician, Seki Takakazu (1642-1708) also discovered Bernoulli numbers at the same time.

1.2 A more modern definition

An equivalent definition of the Bernoulli numbers is obtained from the series expansion of $z/(e^z - 1)$:

$$G(z) = \frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \qquad |z| < 2\pi$$

In other words, the generating function of the Bernoulli number B_k is $z/(e^z - 1)$. The first terms of the expansion of this function are

$$G(z) = (1 + \frac{z}{2!} + \frac{z^2}{3!} + \frac{z^3}{4!} + ...)^{-1} = 1 - \frac{z}{2} + \frac{z^2}{12} - \frac{z^4}{720} + \frac{z^6}{30240} - \frac{z^8}{1209600} + ...$$

which permits to obtain the first value of the Bernoulli numbers :

$$B_0 = 1 \quad , \qquad B_1 = -\frac{1}{2}, \quad B_2 = \frac{1}{6}, \quad B_3 = 0$$
$$B_4 = -\frac{1}{30} \quad , \qquad B_5 = 0, \quad B_6 = \frac{1}{42}, \quad B_7 = 0, \quad B_8 = -\frac{1}{30}$$

We observe that

$$G(z) + \frac{z}{2} = \frac{z}{2}\left(\frac{2}{e^z - 1} + 1\right) = \frac{z}{2}\frac{e^{z/2} + e^{-z/2}}{e^{z/2} - e^{-z/2}} = \frac{z}{2}\coth\frac{z}{2},$$

where coth is the hyperbolic tangent, hence $G(z) + z/2$ is an even function and consequently every Bernoulli numbers of the form B_{2k+1} $(k > 0)$ is null.

Bernoulli polynomials

With a little modification it's possible to define Bernoulli polynomials $B_k(x)$ by

$$G(z, x) = \frac{ze^{zx}}{e^z - 1} = \sum_{k=0}^{\infty} B_k(x)\frac{z^k}{k!}$$

and $G(z, 0) = G(z)$ hence the value of the function $B_k(x)$ at $x = 0$ is B_k and because

$$\frac{\partial G}{\partial x}(z, x) = zG(z, x) = \sum_{k=0}^{\infty} \frac{dB_k}{dx}(x)\frac{z^k}{k!}$$

it follows the important relation

$$\frac{dB_k}{dx}(x) = kB_{k-1}(x).$$

Then, it's easy to deduce that $B_k(x)$ are polynomials of degree k, and the first one are

$$\begin{aligned}
B_0(x) &= 1 \\
B_1(x) &= x - \frac{1}{2} \\
B_2(x) &= x^2 - x + \frac{1}{6} \\
B_3(x) &= x^3 - \frac{3}{2}x^2 + \frac{1}{2}x \\
B_4(x) &= x^4 - 2x^3 + x^2 - \frac{1}{30} \\
&... \\
\end{aligned}$$

Thanks to Bernoulli's polynomials, it's possible to rewrite the expression of the sums $s_p(n)$ as

$$s_p(n) = \sum_{k=0}^{n-1} k^p = \frac{1}{p+1}\left(B_{p+1}(n) - B_{p+1}\right).$$

There are many relations with these polynomials, for example

$$\begin{aligned} B_k(1-x) &= (-1)^k B_k(x), \\ |B_{2k}(x)| &< |B_{2k}| \qquad k = 1, 2, ... \text{and} 0 < x < 1, \\ B_k(\frac{1}{2}) &= -(1-2^{1-k})B_k \qquad k = 0, 1, ... \\ &... \end{aligned}$$

Consult [1] for other formulas.

2 Properties

2.1 Recurrence relation

In the expression

$$s_p(n) = \sum_{k=0}^{p} \frac{B_k}{k!} \frac{p!}{(p+1-k)!} n^{p+1-k}$$

we let $n = 1$, giving

$$0 = \sum_{k=0}^{p} \frac{B_k}{k!} \frac{1}{(p+1-k)!}$$

or equivalently

$$B_p = -\frac{1}{p+1} \sum_{k=0}^{p-1} \binom{p+1}{k} B_k.$$

The last relation allows easy generation of Bernoulli's numbers and shows that the numbers B_p are rational numbers. It convenient to rewrite this relation as the symbolic equation

$$(B+1)^{p+1} - B^{p+1} = 0,$$

and expand the binomial $(B+1)^{p+1}$ where each B^k is replaced by B_k.

Example 1 *With $p = 4$ we have*

$$5B_4 + 10B_3 + 10B_2 + 5B_1 + B_0 = 0$$

thus

$$5B_4 + \left(0 + 10.\frac{1}{6} - 5.\frac{1}{2} + 1\right) = 0 \quad \text{thus} \quad B_4 = -\frac{1}{30}.$$

2.2 Bernoulli's numbers and the zeta function

In 1735, the solution of the *Basel problem* expressed by Jakob Bernoulli some years before, was one of Euler's most sensational discovery. The problem was to find the limit of

$$\zeta(2) = \sum_{n=1}^{\infty} \frac{1}{n^2},$$

he found it to be $\pi^2/6$. He also found the values of the sums

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}}$$

for k up to 13 ([6] and for more details [5]). It's an extraordinary result that $\zeta(2k)$ can be expressed with Bernoulli's Numbers ; the values of these sums are given by

$$\zeta(2k) = \sum_{n=1}^{\infty} \frac{1}{n^{2k}} = \frac{4^k \left|B_{2k}\right| \pi^{2k}}{2(2k)!} \qquad k > 0,$$

(a proof based on two different expansions of $z\,\cot(z)$ is given in [3] p. 383). On the extension of the Zeta function to negative values

$$\zeta(1-2k) = -\frac{B_{2k}}{2k} \qquad k > 0,$$

which may be used to compute B_{2k} (see [4]).

2.3 Asymptotic expansion of Bernoulli's numbers

From the previous relation with the Zeta function, it's clear that

$$\left|B_{2k}\right| = \frac{2(2k)!}{(2\pi)^{2k}}\zeta(2k) \tag{2}$$

and because, when k becomes large, thanks to Stirling's formula

$$\begin{aligned} \zeta(2k) &\sim 1, \\ (2k)! &\sim (2k)^{2k} e^{-2k} \sqrt{4\pi k}, \end{aligned}$$

we have

$$\left|B_{2k}\right| \sim 4 \left(\frac{k}{\pi e}\right)^{2k} \sqrt{\pi k}.$$

In [11], the following results describes how the numerator $N_{2k}$of B_{2k} grows

$$\log\left|N_{2k}\right| = 2k\log(k) + O(k).$$

2.4 Bounds

It may be useful to estimate bounds for B_{2k}, to achieve this we use the following relation between the function $\zeta(s)$ and the alternating series $\zeta_a(s)$

$$\begin{aligned}\zeta(s) &= \frac{\zeta_a(s)}{1-2^{1-s}} \\ \zeta_a(s) &= \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^s}\end{aligned}$$

and since

$$1-\frac{1}{2^s} < \zeta_a(s) < 1,$$

we have the bounds for $\zeta(s)$

$$\frac{1-2^{-s}}{1-2^{1-s}} < \zeta(s) < \frac{1}{1-2^{1-s}}.$$

If we use this relation with (1), the bounds for B_{2k} are therefore

$$\frac{2(2k)!(1-4^{-k})}{(2\pi)^{2k}(1-2.4^{-k})} < |B_{2k}| < \frac{2(2k)!}{(2\pi)^{2k}(1-2.4^{-k})}.$$

3 Clausen von Staudt's theorem

The following important theorem was published in 1840 by Karl von Staudt (1798-1867), it allows to compute easily the fractional part of Bernoulli's numbers (thus it also permits to compute the denominator of those numbers). This theorem was discovered the same year, independently, by Clausen.

Theorem 1 *The value B_{2k}, added to the sum of the inverse of prime numbers p such that $(p-1)$ divides $2k$, is an integer. In other words,*

$$-B_{2k} \equiv \sum_{(p-1)|2k} \frac{1}{p} \pmod 1$$

A complete proof is given in [8], p.91.

When $k > 1$, we observe that the primes $p = 2, 3$ are such as $(p-1)$ divides $2k$. Let's illustrate this theorem with a few examples. For $k = 1$, it becomes

$$\begin{aligned}-B_2 &\equiv \sum_{(p-1)|2} \frac{1}{p} = \frac{1}{2}+\frac{1}{3} = \frac{5}{6} \pmod 1 \\ B_2 &\equiv \frac{1}{6} \pmod 1\end{aligned}$$

for $k = 5$

$$\begin{aligned} -B_{10} &\equiv \sum_{(p-1)|10} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{11} = \frac{61}{66} \pmod 1 \\ B_{10} &\equiv \frac{5}{66} \pmod 1 \end{aligned}$$

and for $k = 8$

$$\begin{aligned} -B_{16} &\equiv \sum_{(p-1)|16} \frac{1}{p} = \frac{1}{2} + \frac{1}{3} + \frac{1}{5} + \frac{1}{17} = \frac{47}{510} \pmod 1 \\ B_{16} &\equiv \frac{463}{510} \pmod 1 \end{aligned}$$

corollary 1 *(due to Rado 1934) For every prime numbers k of the form $3n+1$*

$$B_{2k} \equiv \frac{1}{6} \pmod 1.$$

Proof : It's an easy consequence of the Staudt's theorem since $p-1$ divides $2k = 2(3n+1)$ only if $p-1$ is one of $1, 2, 3n+1, 6n+2$, that is p is one of $2, 3, 3n+2, 6n+3$. But $6n+3$ is divisible by 3 and $3n+2$ is divisible by 2 because $3n+1$ is prime so the only primes p candidates are 2 and 3. •

Example 2 *The first primes of the form $3n+1$ are $7, 13, 19, 31, 37, 43, 61, 67, \ldots$ hence we have*

$$B_{14} \equiv B_{26} \equiv B_{38} \equiv B_{62} \equiv B_{74} \equiv B_{86} \equiv B_{122} \equiv B_{134} \equiv \frac{1}{6} \pmod 1.$$

Clausen von Staudt's theorem also permits to compute exactly a Bernoulli number as soon as a sufficiently good approximation of it is known.

4 Expansion of usual functions

In a previous part we gave the definition

$$\frac{z}{e^z - 1} = \sum_{k=0}^{\infty} B_k \frac{z^k}{k!} \qquad |z| < 2\pi$$

and the consequence

$$\frac{z}{2} \coth \frac{z}{2} = \sum_{k=0}^{\infty} B_{2k} \frac{z^{2k}}{(2k)!}.$$

It obviously leads to the two following expansions

$$\begin{aligned} z\coth(z) &= \sum_{k=0}^{\infty} 4^k B_{2k} \frac{z^{2k}}{(2k)!} = 1 + \frac{z^2}{3} - \frac{z^4}{45} + \frac{2z^6}{945} - \ldots \qquad |z| < \pi \\ z\cot(z) &= \sum_{k=0}^{\infty} (-4)^k B_{2k} \frac{z^{2k}}{(2k)!} = 1 - \frac{z^2}{3} - \frac{z^4}{45} - \frac{2z^6}{945} - \ldots \qquad |z| < \pi \end{aligned}$$

where $\cot(z) = \cos(z)/\sin(z) = i\coth(iz)$ is the cotangent function. Now it's possible to find the expansion for $\tanh(z)$ and $\tan(z)$, if we observe that

$$2\coth(2z) - \coth(z) = 2\frac{\cosh(2z)}{\sinh(2z)} - \frac{\cosh(z)}{\sinh(z)} = \frac{\cosh^2(z) + \sinh^2(z)}{\sinh(z)\cosh(z)} - \frac{\cosh(z)}{\sinh(z)} = \tanh(z)$$

so

$$\begin{aligned} \tanh(z) &= \sum_{k=1}^{\infty} 4^k (4^k - 1) B_{2k} \frac{z^{2k-1}}{(2k)!} = z - \frac{z^3}{3} + \frac{2z^5}{15} - \frac{17z^7}{315} + \ldots \qquad |z| < \frac{\pi}{2} \\ \tan(z) &= \sum_{k=1}^{\infty} (-4)^k (1 - 4^k) B_{2k} \frac{z^{2k-1}}{(2k)!} = z + \frac{z^3}{3} + \frac{2z^5}{15} + \frac{17z^7}{315} + \ldots \qquad |z| < \frac{\pi}{2}. \end{aligned}$$

Bernoulli's numbers also occur in the expansions of

$$\frac{z}{\sin(z)}, \frac{z}{\sinh(z)}, \log(\frac{\sin(z)}{z}), \log(\cos(z)), \log(\frac{\tan(z)}{z}), \ldots$$

4.1 Series

Setting $z = 1$ or $z = -1$ in the expansion of $z/(e^z - 1)$ lead to the series

$$\begin{aligned} \frac{1}{e-1} &= \sum_{k=0}^{\infty} \frac{B_k}{k!} = \frac{1}{2} + \sum_{k=1}^{\infty} \frac{B_{2k}}{(2k)!} \\ \frac{e}{e-1} &= \sum_{k=0}^{\infty} \frac{(-1)^k B_k}{k!} \end{aligned}$$

and

$$\frac{B_{2k}}{(2k)!} \sim \frac{2}{(4\pi^2)^k} \approx \frac{2}{39.48^k}.$$

5 Euler-Maclaurin formula

Let $f(x)$ be a function of class C^{2p+2} on an interval $[a, b]$ and let $h = (b-a)/m$ a subdivision of this interval into m equal parts then we have the important result

Theorem 2 *There exist* $0 < \vartheta < 1$ *and*

$$\sum_{k=0}^{m} f(a+kh) = \frac{1}{h}\int_a^b f(x)dx + \frac{1}{2}\left(f(a)+f(b)\right) + \sum_{k=1}^{p} \frac{h^{2k-1}}{(2k)!} B_{2k}\left(f^{(2k-1)}(b) - f^{(2k-1)}(a)\right)$$
$$+ \frac{h^{2p+2}}{(2p+2)!} B_{2p+2} \sum_{k=0}^{m-1} f^{(2p+2)}(a+kh+\vartheta h).$$

A proof is given in [7].

This formula was first studied by Euler in 1732 and independently by Colin Maclaurin (1698-1746) in 1742 [10]. Euler used it to compute sums of slow converging series and Maclaurin used it as a numerical quadrature formula.

With the same conditions, setting $n = m+1, a = 1, b = n, h = 1$ the theorem becomes :

Theorem 3

$$\sum_{k=1}^{n} f(k) = \int_1^n f(x)dx + \frac{1}{2}\left(f(1)+f(n)\right) + \sum_{k=1}^{p} \frac{B_{2k}}{(2k)!}\left(f^{(2k-1)}(n) - f^{(2k-1)}(1)\right) + R_n(f,p),$$

where $R_n(f,p)$ *is the remainder bounded by*

$$R_n(f,p) \leq \frac{2}{(2\pi)^{2p}} \int_1^n \left| f^{(2p+1)}(x) \right| dx$$

5.1 Applications

1. $f(x) = x^2$, the remainder is null since $f^{(p)}(x) = 0$ for $p > 2$:

$$\sum_{k=1}^{n} k^2 = \int_1^n x^2 dx + \frac{1}{2}\left(1+n^2\right) + \frac{B_2}{2}(2n-2) + 0$$
$$= \frac{n^3-1}{3} + \frac{1+n^2}{2} + \frac{n-1}{6} = \frac{n(n+1)(2n+1)}{6}.$$

2. $f(x) = 1/x$, $f^{(2k-1)}(x) = -(2k-1)!/x^{2k}$, Euler-Maclaurin formula yields for a given p :

$$\sum_{k=1}^{n} \frac{1}{k} - \log(n) = \frac{1}{2} + \frac{1}{2n} + \sum_{k=1}^{p} \frac{B_{2k}}{2k}\left(1 - \frac{1}{n^{2k}}\right) + R_n(f,p)$$

when $n \to \infty$, the left hand side of the equality tends to γ (Euler's constant) and the equality gives

$$\gamma = \frac{1}{2} + \sum_{k=1}^{p} \frac{B_{2k}}{2k} + R_\infty(f,p),$$

finally

$$\sum_{k=1}^{n} \frac{1}{k} - \log(n) = \gamma + \frac{1}{2n} - \sum_{k=1}^{p} \frac{B_{2k}}{2k} \frac{1}{n^{2k}} + (R_n(f,p) - R_\infty(f,p)) .$$

(check that $R_n(f,p) - R_\infty(f,p) = O(1/n^{2p+2})$)

3. $f(x) = \log(x)$, with the same method (left as exercise) Euler-Maclaurin formula becomes

$$\sum_{k=1}^{n} \log(k) = n\log(n) - n + \frac{\log(n)}{2} + \log(\sqrt{2\pi}) + \sum_{k=1}^{p} \frac{B_{2k}}{2k(2k-1)} \frac{1}{n^{2k-1}} + O\left(\frac{1}{n^{2p+1}}\right)$$

so, for example, with $p = 3$

$$\log(n!) = n\log(n) - n + \log(\sqrt{2\pi n}) + \frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} + O\left(\frac{1}{n^7}\right)$$

and taking the exponential

$$n! = n^n e^{-n} \sqrt{2\pi n} \exp\left(\frac{1}{12n} - \frac{1}{360n^3} + \frac{1}{1260n^5} + O\left(\frac{1}{n^7}\right)\right) .$$

This is the *asymptotic Stirling formula.* Using the series expansion of the exponential function near the origin, it's more convenient to write it as

$$n! = n^n e^{-n} \sqrt{2\pi n} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + \frac{163879}{209018880n^5} + O\left(\frac{1}{n^6}\right)\right)$$

6 Bernoulli's Numbers and Fermat's theorem

The famous *Fermat's Last theorem* states that the equation

$$x^n + y^n = z^n$$

never has non-zero integer solutions for $n > 2$. Since Fermat expressed this result around 1630, the pursuit of a proof occupied generations of mathematicians. A big step was made by Kummer in 1850 when he proved Fermat's Theorem for $n = p$, whenever p is, what is called today, a *regular prime.* Kummer gave the beautiful regularity criterion :

p isaregularprimeifandonlyif p doesnotdividethenumeratorof $B_2, B_4, ..., B_{p-3}$.

He showed that all primes before 37 where regular, hence Fermat's theorem was proved for those primes. 37 is the first non regular prime because it divides the numerator of

$$B_{32} = -\frac{7709321041217}{510} = -\frac{208360028141 \times 37}{510}$$

The next irregular primes (less than 300) are

$$59, 67, 101, 103, 131, 149, 157, 233, 257, 263, 271, 283, 293, ...$$

For example 157 divides the numerators of B_{62} and B_{110}.

Thanks to arithmetical properties of Bernoulli's numbers, Jensen proved, in 1915, that the number of irregular primes is infinite. Even if it's probable that the number of regular primes is infinite, the proof remains unknown [11].

7 The first Bernoulli Numbers

$B_{2k+1} = 0$, for $k > 0$ and

$$\begin{aligned}
B_0 &= 1, B_1 = -1/2, B_2 = 1/6, B_4 = -1/30, B_6 = 1/42, B_8 = -1/30, B_{10} = 5/66, \\
B_{12} &= -691/2730, B_{14} = 7/6, B_{16} = -3617/510, B_{18} = 43867/798, B_{20} = -174611/330, \\
B_{22} &= 854513/138, B_{24} = -236364091/2730, B_{26} = 8553103/6, B_{28} = -23749461029/870, \\
B_{30} &= 8615841276005/14322, B_{32} = -7709321041217/510, B_{34} = 2577687858367/6, ...
\end{aligned}$$

More numbers are given in [12] and in [1].

Euler computed these numbers up to B_{30}, then Ohm extended the calculation up to B_{62} in 1840. A few years later, in 1877, Adams made the impressive computation of Bernoulli's numbers up to B_{124}(or B^*_{62} according to his convention)[2]. The numerator of B_{124} has 110 digits and the denominator is the number 30.

In 1996, Simon Plouffe and Greg J. Fee computed B_{200000}a huge number of about 800000 digits, the computation took about 2 hours on a work station.

References

[1] M. Abramowitz and I. Stegun, *Handbook of Mathematical Functions*, Dover, New York, (1964)

[2] J.C. Adams, *On the calculation of Bernoulli's numbers up to $B_{62}$by means of Staudt's theorem*, Rep. Brit. Ass., (1877)

[3] J.M. Borwein and P.B. Borwein, *Pi and the AGM - A study in Analytic Number Theory and Computational Complexity*, A Wiley-Interscience Publication, New York, (1987)

[4] P. Borwein, *An efficient algorithm for the Riemann Zeta function*, (1995)

[5] W. Dunham, *Euler The Master of Us All*, The Mathematical Association of America, (1999)

[6] L. Euler, *Introduction à l'analyse infinitésimale* (french traduction by Labey), Barrois, ainé, Librairie, (original 1748, traduction 1796), vol. 1

[7] R.L. Graham, D.E. Knuth and O. Patashnik, *Concrete Mathematics*, Addison-Wesley, (1994)

[8] G.H. Hardy and E. M. Wright, *An Introduction to the Theory of Numbers*, Oxford Science Publications, (1979)

[9] V. J. Katz, *A History of Mathematics-An Introduction*, Addison-Wesley, (1998)

[10] C. Maclaurin, *A Treatise of fluxions*, Edinburgh, (1742)

[11] P. Ribenboim, *The new Book of Prime Number Records*, Springer, (1996)

[12] L. Saalschütz, *Vorlesungen über die Bernoullischen Zahlen*, Berlin, Verlag von Julius Springer, (1893)